

Maintaining Approximate Maximum Weighted Matching in Fully Dynamic Graphs

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Abstract

We present a fully dynamic algorithm for maintaining approximate maximum weight matching in general weighted graphs. The algorithm maintains a matching M whose size is at least $\frac{1}{8}M^*$ where M^* is the weight of the maximum weight matching. The algorithm achieves an expected amortized $O(\log n \log \mathcal{C})$ update time, where \mathcal{C} is the ratio between the maximum and the minimum weights of edges in the given graph. Using a simple randomized scaling technique, we are able to obtain a matching whose expected size is at least $\frac{1}{6}M^*$.

1 Introduction

Given an undirected graph $G = (V, E)$, where V represents the set of vertices and E represents the set of edges in the graph, let $n = |V|$ and $m = |E|$. In a weighted graph, we associate a weight function $w : E \rightarrow \mathbb{R}$ where $\forall e \in E$, $w(e)$ represents the weight of the edge. The weight function for a set of edges $M \subseteq E$ is represented by $w(M)$ and defined as $\sum_{e \in M} w(e)$.

A *matching* in a graph is a set of independent edges in the graph. Thus, a subset M of E is called a matching if no vertex of the graph is incident on more than one edge in M . In an unweighted graph, we refer maximum matching as maximum cardinality matching(MCM) while in a weighted graph we refer to it as a maximum weight matching(MWM). A matching is called α -MWM(α -MCM) if it is at least $\frac{1}{\alpha}$ factor of MWM(MCM), and is represented by α -MWM(α -MCM).

A dynamic graph algorithm aims to answer queries regarding some property of the graph which is undergoing a series of changes due to insertion or deletion of edges (V is assumed to be fixed). The updates can be classified naturally into two types - insertions and deletions. Algorithms which handle only insertions are called incremental and the ones that can deal only with deletions are called decremental. An algorithm that can handle both insertions and deletions of edges is called a *fully dynamic* algorithm. In this paper, we address the problem of maintaining approximate weighted matching in a fully dynamic graph.

Previous Results

The fastest known algorithm for finding MCM in general graphs is by Micali and Vazirani[7] that runs in $O(m\sqrt{n})$ time. Their algorithm can be used to compute a matching having size $(1 - \epsilon)$ times the size of maximum matching in $O(m/\epsilon)$ time. Mucha and Sankowski[8] gave an algorithm that computes MCM in $O(n^\omega)$, where $\omega < 2.376$ is the exponent of n in the fastest known matrix multiplication algorithm. Relatively few algorithms are known that maintain matching in dynamic graphs. The first result was presented by Ivkovic and Lloyd[5] with amortized update time $O(n + m)^{0.7072}$. Onak and Rubinfeld[9] achieved polylogarithmic amortized expected update time and maintained an α -approximate MCM where α was shown to be some constant but not explicitly calculated. Baswana, Gupta and Sen[1] presented a fully dynamic randomized algorithm for maintaining maximal¹ matching in expected amortized $O(\log n)$ update time.

For MWM, Gabow[4] gave an $O(mn + n^2 \log n)$ time algorithm. Preis[10] gave a $O(m)$ time algorithm for finding 2-MWM. Drake and Hougardy[2] gave a simpler algorithm for the same problem. Vinkemeier and Hougardy[11] presented an algorithm to get a matching which is $(2/3 - \epsilon)$ times the size of MWM in $O(m/\epsilon)$ time. Duan, Pettie and Su[3] presented an algorithm to find a matching which is $(1 - \epsilon)$ times the size of MWM in $O(m\epsilon^{-1} \log \epsilon^{-1})$ time. To the best of our knowledge, there have been no significant results for maintaining MWM or approximate MWM in dynamic graphs.

Preliminaries

Let M be the set of matched edges in the graph. A vertex is called *free* if it is not incident on any edge in M . A vertex which is not free is called *matched*. Similarly, an edge is called *matched* if it is in M and is called *free* otherwise. Suppose (u, v) is a matched edge, then u is said to be the *mate* of v and vice versa. A matching M is called *maximal* if no edge can be added to the matching without violating the degree bound of one for a matched vertex. An alternating path is defined as a path in which edges are alternately matched and free, while an augmenting path is an alternating path which begins and ends with free vertices.

Our Results

We present a fully dynamic algorithm that achieves expected amortized $O(\log n \log \mathcal{C})$ update time for maintaining 8-MWM. Subsequently, we improve this approximation ratio to 6 using a simple randomized scaling technique. Here \mathcal{C} is the ratio of the highest weight edge to the smallest weight edge in the given graph. It uses the result of Baswana, Gupta and Sen [1] as a subroutine. The algorithm uses some ideas from an approximate (distributed) algorithm given by Lotker, Patt-Shamir, and Rosen[6]. They give a distributed algorithm having running time $O(1)$ per update and which maintains a 27-MWM in a dynamic weighted graph. The first subsection describes the basic ideas and the intuition behind the algorithm. Section 2 describes our algorithm.

Overview of our approach

The basic idea of the algorithm is to try and reduce the problem of finding an approximate maximum weight matching to the problem of finding approximate matching in unweighted

¹a 2-MCM

graphs. Observe that finding MCM is a special case of finding MWM with all edges having the same weight. So, a maximal matching is 2-MWM when all edges have the same weight. But this observation does not immediately extend to general the weights. First, let us consider the case when the weights are within a range, say, $[\alpha^i, \alpha^{i+1})$, where $\alpha > 1$ is a constant. In such a graph the maximal matching gives a 2α approximation of the maximum weight matching. This follows directly from the fact that maximal matching in an undirected graph gives 2 approximation of maximum matching. So, maximal matching can be used as an approximation for MWM in a graphs where the weights of the edges are bounded. To generalize this, we partition the edges of the graph into levels according to their weight. Edges at level i have weights in the range $[\alpha^i, \alpha^{i+1})$ and the set of edges at level i is represented by E_i , viz., $\forall e \in E_i, w(e) \in [\alpha^i, \alpha^{i+1})$.

Observe that in this scheme of partitioning, any edge is present only at one level while a vertex may have edges incident on it from various levels. The subgraph at level i is represented by $G_i = (V, E_i)$ and a maximal matching M_i is maintained for G_i . The maximal matching at each level provides an approximation for the maximum matching at that level. To obtain an overall bound on MWM, the approximations obtained by the maximal matchings from different levels have to be *combined* in some way. The need for this combining arises from our earlier observation that even if only the edges in $\cup_i M_i$ are considered, a vertex may have multiple edges incident on it. So, some of the edges have to be removed from $\cup_i M_i$ to find a valid matching of G . Let $\mathcal{H} = (V, \cup M_i)$ be the subgraph of G having only those edges which are part of the maximal matching at some level. In this subgraph, a matching is maintained which is shown to be 8-MWM.

2 Fully Dynamic 8-MWM

We maintain a partition of edges according to their levels. A maximal matching M_i is maintained at each level. We use the fully dynamic algorithm in [1] to maintain M_i 's. Their algorithm will cause edges to become matched from free and vice-versa leading to addition or deletion of edges to the subgraph \mathcal{H} . They show that there is an algorithm which can maintain maximal matching in a dynamic graph in $O(\log n)$ amortized time with high probability. This means that the amortized number of matched edges added or removed from the subgraph \mathcal{H} is $O(\log n)$ at each update step. We maintain the output matching \mathcal{M} in this subgraph \mathcal{H} taking advantage of the hierarchical structure of \mathcal{H} . Since \mathcal{H} is formed by the union of matchings at various levels, a vertex can have at most one neighbor at each level. The matching is maintained such that for every edge of \mathcal{H} which is not in \mathcal{M} there is an edge adjacent to it at a higher level which is in \mathcal{M} . For an edge e , let $Level(e)$ denote its level. Formally the *invariant* is,

$\forall e \in E(\mathcal{H}),$ either $e \in \mathcal{M}$ or e is adjacent to an edge $e' \in \mathcal{M}$ such that $Level(e') > Level(e)$.

Data Structures Used

The algorithm requires information to be maintained about the graph and its vertices and edges. We explicitly show how the following are maintained -

- M_l - A maximal matching at the level l .

- $Free(v)$ - Whether the vertex v is free in the matching \mathcal{M} .
- $Mate(v)$ - The mate of v , if it is not free.
- $Level((u, v))$ or $Level(e)$ - The level at which the edge e or the edge (u, v) is according to the condition that $\forall e \in G_i, w(e) \in [\alpha^i, \alpha^{i+1})$.
- $OccupiedLevels$ - The set of levels where there is at least one edge from \mathcal{H} .
- L^{max} - The highest occupied level.
- L^{min} - The lowest occupied level.
- $N(v, l)$ - It maintains the neighbor of v at the level l if there is one, *null* otherwise.
- \mathcal{M} - The matching being maintained.

The next subsection describes a static algorithm for computing \mathcal{M} in the graph \mathcal{H} . The static algorithm gives way to a dynamic algorithm described in the subsection following it.

Static Algorithm to obtain \mathcal{M} from \mathcal{H}

Procedure 2.1: StaticCombine()

```

1  $\mathcal{M} = \phi;$ 
2 for  $i = L^{max}$  to  $L^{min}$  do
3    $\mathcal{M} = \mathcal{M} \cup M_i;$ 
4   for  $(u, v) \in M_i$  do
5     for  $j = i - 1$  to  $L^{min}$  do
6       for  $(x, y) \in M_j$  do
7         if  $u = x$  or  $u = y$  or  $v = x$  or  $v = y$  then
8            $M_j = M_j \setminus \{(x, y)\};$ 

```

The static algorithm divides the edges from the graph G into levels and a maximal matching M_i is obtained for each of the levels. Using these maximal matchings we get the graph \mathcal{H} . Thereafter the level numbers L^{max} and L^{min} are computed and the procedure **StaticCombine** is used.

The procedure **StaticCombine** starts by picking all the edges in \mathcal{H} at the highest level and adds them to the matching \mathcal{M} . For every edge (u, v) added to the matching \mathcal{M} , all the edges in the graph \mathcal{H} incident on u and v have to be removed from the graph. The same process is repeated for the next lower level. Note that every edge in \mathcal{H} is either in the matching \mathcal{M} or is deleted due to the addition of a neighboring edge in \mathcal{M} and thus maintains the invariant. Also, the matching \mathcal{M} is maximal in \mathcal{H} because of the way it is being computed.

Dynamic Algorithm to maintain \mathcal{M}

Before any addition or deletion of edge, we assume that there was a matching \mathcal{M} which satisfied the invariant. We begin the description of the algorithm in terms of addition and deletion of edges to the graph \mathcal{H} . The addition and deletion of the edges in \mathcal{H} is caused due to addition/deletion of an edge in G . We describe some of the frequently used procedures first. Then the procedures for addition and deletion of edges in \mathcal{H} and finally the procedures for addition and deletion of edges in G are described.

Procedure 2.2: AddToMatching(u, v)

```

1  $Free(u) = False; Free(v) = False;$ 
2  $Mate(u) = v; Mate(v) = u;$ 
3  $\mathcal{M} = \mathcal{M} \cup \{(u, v)\};$ 

```

Procedure 2.3: DelFromMatching(u, v)

```

1  $Free(u) = True; Free(v) = True;$ 
2  $\mathcal{M} = \mathcal{M} \setminus \{(u, v)\};$ 

```

The procedure **AddToMatching** adds an edge to the matching updating the free and mate fields accordingly. The procedure **DelFromMatching** deletes an edge from the matching updating the mate and the free fields correctly. Both of them execute in $O(1)$ time.

Procedure 2.4: HandleFree(u, lev)

```

1 for  $l$  from  $lev$  to  $L^{min}$  do
2    $v = N(u, l);$ 
3   if  $v$  is not null then
4     if  $v$  is free then
5       AddToMatching ( $u, v$ );
6       return;
7     else if  $Level((v, Mate(v))) < l$  then
8        $v' = Mate(v);$ 
9       DelFromMatching ( $v, v'$ );
10      AddToMatching ( $u, v$ );
11      HandleFree ( $v', Level((v, v'))$ );
12      return;

```

The procedure **HandleFree** takes as an input a vertex u which has become free and a level number lev from where it has to start looking for a mate. The procedure assumes that u cannot find a mate at any level above lev . The procedure checks the neighbors of the vertex u at all levels below lev falling step by step. If it finds a free neighbor the corresponding edge is added to the matching and thereafter nothing is done. Otherwise if a vertex is found which has a mate at a lower level then we are violating the invariant as (u, v) does not belong to \mathcal{M} and is neighboring to an edge in \mathcal{M} at a lower level. So, the edge $(v, Mate(v))$ is removed

from the matching and the edge (u, v) is added to the matching. This change results in a free vertex which is at a lower level which has to be handled. Note that the recursive calls to **HandleFree** are all with lower level numbers. So, the procedure takes $O(L^{max} - L^{min})$ time.

Procedure 2.5: AddEdge(u, v)

```

1  $l = Level((u, v));$ 
2  $N(u, l) = v;$ 
3  $N(v, l) = u;$ 
4 if  $u$  is free and  $v$  is free then
5   | AddToMatching ( $u, v$ );
6 else if  $u$  is free and  $v$  is not free then
7   | if  $Level((u, Mate(u))) < l$  then
8     |    $v' = Mate(v);$ 
9     |   DelFromMatching ( $v, v'$ );
10    |   AddToMatching ( $u, v$ );
11    |   HandleFree ( $v', Level((v, v'))$ );
12 else if  $u$  is not free and  $v$  is free then
13   | if  $Level((v, Mate(v))) < l$  then
14     |    $u' = Mate(u);$ 
15     |   DelFromMatching ( $u, u'$ );
16     |   AddToMatching ( $u, v$ );
17     |   HandleFree ( $u', Level((u, u'))$ );
18 else if  $Level((v, Mate(v))) < l$  and  $level((u, Mate(u))) < l$  then
19   |  $u' = Mate(u); v' = Mate(v);$ 
20   | DelFromMatching ( $u, u'$ ); DelFromMatching ( $v, v'$ );
21   | AddToMatching ( $u, v$ );
22   | HandleFree ( $u', Level((u, u'))$ );
23   | HandleFree ( $v', Level((v, v'))$ );

```

The procedure **AddEdge** handles addition of edges to \mathcal{H} . Suppose the edge (u, v) is added to \mathcal{H} . It is checked whether both u and v are free or not. If so the edge (u, v) is added to the matching \mathcal{M} . If not then at least one of u and v is matched. It checks for the edge (u, v) whether it is adjacent to an edge in \mathcal{M} which is at a higher level or not. If (u, v) is adjacent to a higher level edge in \mathcal{M} , then nothing is done else (u, v) is adjacent to lower level edge(s) in \mathcal{M} , thus violating the invariant. The procedure removes the lower level edge(s) from the matching and adds the edge (u, v) to the matching. At most 2 vertices can become free due to the addition of this edge which are handled using the procedure **HandleFree**. If u' was the previous mate of u , the the edge (u, u') is removed from \mathcal{M} . Since \mathcal{M} satisfied the invariant before addition of this edge, all the neighboring edges of u' at higher level than $Level(u, u')$ are matched to a vertex at higher levels. So u' has to start looking for mates from the level of (u, u') . The procedure makes a constant number of calls to **HandleFree** and thus runs in $O(L^{max} - L^{min})$ time.

The procedure **DeleteEdge** does nothing if an unmatched edge is deleted. If a matched edge (u, v) is deleted at level l , it calls **HandleFree** for both the end points to restore the

Procedure 2.6: DeleteEdge(u, v)

```
1  $l = \text{Level}((u, v));$ 
2  $N(u, l) = \text{null};$ 
3  $N(v, l) = \text{null};$ 
4 if  $(u, v) \in \mathcal{M}$  then
5   |  $\text{DelFromMatching}(u, v);$ 
6   |  $\text{HandleFree}(u, l);$ 
7   |  $\text{HandleFree}(v, l);$ 
```

invariant in the matching. **HandleFree** is called with the level l because our invariant implies that all the neighbors of u and v are matched at higher levels. So they cannot find a mate at higher levels. This again takes $O(L^{\max} - L^{\min})$ time.

Procedure 2.7: EdgeUpdate(u, v, type)

```
1  $l = \text{Level}((u, v)) = \lfloor \log_{\alpha} w(u, v) \rfloor;$ 
2 if  $\text{type}$  is addition and  $M_l$  is  $\phi$  then
3   |  $\text{OccupiedLevels} = \text{OccupiedLevels} \cup \{l\};$ 
4   | Update  $L^{\max}$  and  $L^{\min};$ 
5 Update  $M_l$  using the algorithm in [1];
6 if  $\text{type}$  is deletion and  $M_l$  is  $\phi$  then
7   |  $\text{OccupiedLevels} = \text{OccupiedLevels} \setminus \{l\};$ 
8   | Update  $L^{\max}$  and  $L^{\min};$ 
9 Let  $\mathcal{D}$  be the set of edges deleted from  $M_l$  in step 5;
10 Let  $\mathcal{A}$  be the set of edges added to  $M_l$  in step 5;
11 for  $(x, y) \in \mathcal{D}$  do
12   |  $\text{DeleteEdge}(x, y);$ 
13 for  $(x, y) \in \mathcal{A}$  do
14   |  $\text{AddEdge}(x, y);$ 
```

The function **EdgeUpdate** handles addition and deletion of edge in G . It finds out the level of the edge and updates the maximal matching at that level. It updates the **OccupiedLevels** set accordingly. This set is required because the values of L^{\max} and L^{\min} are to be maintained. The algorithm in [1] can be easily augmented to return the set of edges being added or deleted from the maximal matching in each update. As discussed before, amortized $O(\log n)$ edges change their status per update. So, the procedure and the algorithm has an expected amortized update time of $O(\log n \cdot (L^{\max} - L^{\min}))$. Let e^{\max} and e^{\min} represent the edges having the maximum and the minimum weight in the graph. Recall that $\mathcal{C} = w(e^{\max})/w(e^{\min})$.

$$L^{\max} - L^{\min} < \log_{\alpha} w(e^{\max}) - \log_{\alpha} w(e^{\min}) + 1 = O\left(\log \frac{w(e^{\max})}{w(e^{\min})}\right) = O(\log \mathcal{C})$$

So we can claim that

Claim 2.1. *The expected amortized update time of the algorithm per edge insertion or deletion is $O(\log n \log \mathcal{C})$.*

In the next section we analyze the algorithm to prove that there is a choice of α for which the approximation factor becomes 8.

2.1 Analysis

To get a good approximation ratio, we bound the weight of M^* with the weight of \mathcal{M} . We now state a few simple observations which help in understanding the analysis.

Observation 2.1. *Since M^* is a matching, no two edges of M^* can be incident on the same vertex.*

Observation 2.2. *For any edge $e \notin M^*$, there can be at most two edges of M^* which are adjacent to e , one for each endpoint of e .*

To bound the weight of M^* using the weight of \mathcal{M} , we define a many to one mapping $\phi: E \rightarrow E$. This mapping maps every edge in M^* to an edge in \mathcal{M} . Using this mapping, we find out all the edges which are mapped to an edge $e \in \mathcal{M}$ and bound their weight using the weight of e . Let this set be denoted by $\phi^{-1}(e)$. For an edge $e \in M^*$, the mapping is defined as:

1. If $e \in E(\mathcal{H})$ and $e \in \mathcal{M}$ then $\phi(e) = e$.
2. If $e \in E(\mathcal{H})$ and $e \notin \mathcal{M}$ then our invariant ensures that e is adjacent to an edge e' such that $\text{Level}(e') > \text{Level}(e)$, then $\phi(e) = e'$. If e is adjacent to two matched edges in \mathcal{M} , map e to any one of them. As a rule, if two edges are available for mapping, then we will map e to any one of them.
3. If $e \notin E(\mathcal{H})$, then e is adjacent to an edge $e' \in E(\mathcal{H})$, if $e' \in \mathcal{M}$ then $\phi(e) = e'$.
4. If $e \notin E(\mathcal{H})$, then e is adjacent to an edge $e' \in E(\mathcal{H})$, if $e' \notin \mathcal{M}$ then e' is adjacent to an edge $e'' \in \mathcal{M}$ such that $\text{Level}(e'') > \text{Level}(e')$, $\phi(e) = e''$.

Now that we have defined a many to one mapping, we find out the edges of M^* which are mapped to an edge $e \in \mathcal{M}$. An edge which is mapped to e can either be adjacent to e or e itself or not adjacent to e . If an edge of M^* , which is mapped to $e \in \mathcal{M}$, is adjacent to e or is e itself, then it is called a *Directly mapped edge*. An edge of M^* which is mapped to $e \in \mathcal{M}$ and is not adjacent to e is called an *Indirectly mapped edge*. Let $\phi_D^{-1}(e)$ and $\phi_I^{-1}(e)$ be the set of directly mapped and indirectly mapped edges respectively for an edge $e \in \mathcal{M}$. Directly mapped edges are of type 1, 2 and 3 as described above and indirectly mapped edges are of type 4. Since we maintain a maximal matching \mathcal{M} , this ensures that an edge $e \in M^*$ is in \mathcal{M} or is adjacent to an edge in \mathcal{M} . This ensures that all the edges in M^* are mapped by ϕ .

A directly mapped edge maybe of the first type as described above. If an edge $e \in \mathcal{M}$ has a directly mapped edge of type 1, it will not have any other directly mapped edge. This follows from the definition of a directly mapped edge and Observation 2.1. There can be at most two directly mapped edges of the second type (Observation 2.2). These edges mapped to e are always from a level $< \text{Level}(e)$. There can be at most two directly mapped edges of type 3 also if they are not in \mathcal{H} but are adjacent to e . By Observation 2.2, there can only be two such edges.

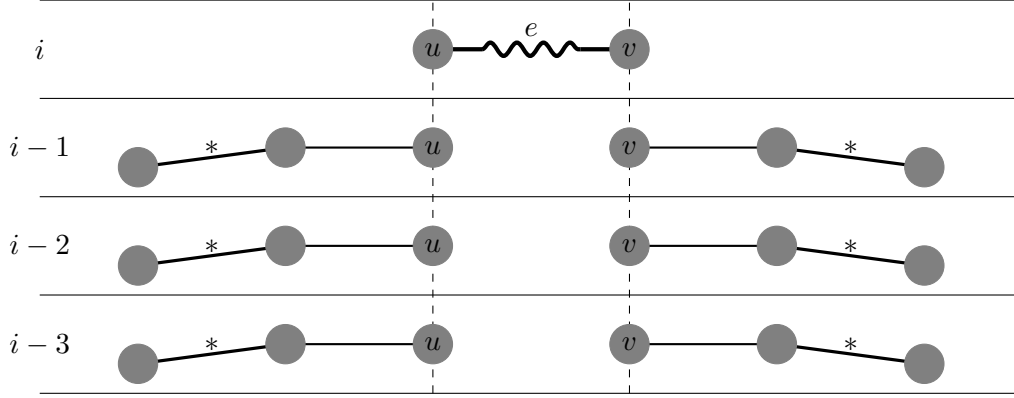


Figure 1: $e \in \mathcal{M}$. The edges marked $*$ are not in \mathcal{H} and are in M^* . The edges which are not marked $*$ are all in \mathcal{H} . All the edges marked by $*$ are indirectly mapped to e .

Claim 2.2. *There can be at most two directly mapped edges to an edge $e \in \mathcal{M}$ at any level.*

The total weight of the edges directly mapped to e will be maximum when both of them are from the same level as e . Assume that e is at level i . Summing the weights of the edges which are directly mapped to e , we get

$$\sum_{e^* \in \phi_D^{-1}(e)} w(e^*) < 2 * \alpha^{i+1} < 2\alpha w(e) \quad (1)$$

Indirectly mapped edges can only be of the fourth kind in which the edge is not in \mathcal{H} , but is adjacent to an edge in \mathcal{H} , which in turn is adjacent to e . By definition, these edges are from a level lower than that of e . There can be at most two edges from each level lower than $\text{Level}(e)$ which are in \mathcal{H} and are adjacent to e (see Figure 1).

Claim 2.3. *There can be at most two indirectly mapped edges to an edge $e \in \mathcal{M}$ at level $< \text{Level}(e)$.*

Note that there can be a large number of edges which are indirectly mapped to e . Still we will be able to get a good bound on their total weight. This is because there can be at most two indirectly mapped edges from each level and the weight of edges in the levels decreases geometrically as we go to lower levels.

Assume that e is at level i . Summing the weight of edges which are indirectly mapped to e , we get

$$\sum_{e^* \in \phi_I^{-1}(e)} w(e^*) < 2 \sum_{j=i-1}^{L^{min}} \alpha^{j+1} < \frac{2\alpha^{i+1}}{\alpha-1} < \frac{2\alpha w(e)}{\alpha-1} \quad (2)$$

Thus, the total weight mapped to e is -

$$\sum_{e^* \in \phi^{-1}(e)} w(e^*) = \sum_{e^* \in \phi_D^{-1}(e)} w(e^*) + \sum_{e^* \in \phi_I^{-1}(e)} w(e^*) < w(e) \left(\frac{2\alpha}{\alpha-1} + 2\alpha \right)$$

As reasoned before, an edge in M^* is mapped to some edge in \mathcal{M} . So summing this over all the edges in \mathcal{M} , we get

$$\sum_{e \in \mathcal{M}} w(e) \left(\frac{2\alpha}{\alpha-1} + 2\alpha \right) > \sum_{e \in \mathcal{M}} \sum_{e^* \in \phi^{-1}(e)} w(e^*) = \sum_{e^* \in M^*} w(e^*)$$

The function $f(\alpha) = \left(\frac{2\alpha}{\alpha-1} + 2\alpha\right)$ attains its minimum value of 8 at $\alpha = 2$. So, if the value of α is picked to be 2, we get an 8 approximation maximum weight matching algorithm. We can state the following theorem.

Theorem 2.1. *There exists a fully dynamic algorithm that maintains 8-MWM for any graph on n vertices in expected amortized $O(\log n \log \mathcal{C})$ time per update.*

3 Improvements

If e is an edge in \mathcal{M} at level i , then the approximation ratio calculated in the previous section can be written as

$$\begin{aligned} \frac{1}{w(e)} \sum_{e^* \in \phi^{-1}(e)} w(e^*) &= \frac{1}{w(e)} \left(\sum_{e^* \in \phi_D^{-1}(e)} w(e^*) + \sum_{e^* \in \phi_I^{-1}(e)} w(e^*) \right) \\ &\leq \sum_{e^* \in \phi_D^{-1}(e)} \frac{w(e^*)}{2^i} + \sum_{e^* \in \phi_I^{-1}(e)} \frac{w(e^*)}{2^i} \quad (w(e) \geq 2^i) \\ &= \left(\sum_{\substack{e^* \in \phi_D^{-1}(e) \\ \text{Level}(e^*)=i}} \frac{w(e^*)}{2^i} \right) + \sum_{j=1}^{i-1} \frac{1}{2^{i-j}} \sum_{\substack{e^* \in \phi_D^{-1}(e) \\ \text{Level}(e^*)=j}} \frac{w(e^*)}{2^j} \end{aligned}$$

For every edge e^* at level i , if $w(e^*)/2^i < \gamma$, then the approximation ratio is calculated as

$$\begin{aligned} \frac{1}{w(e)} \sum_{e^* \in \phi^{-1}(e)} w(e^*) &\leq \left(\sum_{\substack{e^* \in \phi_D^{-1}(e) \\ \text{Level}(e^*)=i}} \gamma \right) + \sum_{j=1}^{i-1} \frac{1}{2^{i-j}} \sum_{\substack{e^* \in \phi_D^{-1}(e) \\ \text{Level}(e^*)=j}} \gamma \\ &= 2\gamma + \sum_{j=1}^{i-1} \frac{1}{2^{i-j}} \cdot 2\gamma \leq 4\gamma \quad (\text{By Claim 2.2 and 2.3}) \end{aligned}$$

In the previous section, $\gamma = 2$ in the worst case, so we get 8-approximation ratio. To improve the approximation ratio, we use randomization to find a good expected value of γ . To do this, we multiply all the weight of edges with a number chosen uniformly randomly from $[1, 2]$. Since we are multiplying each edge with the same random number, M^* does not change. Consider an edge e at level i having weight $w(e) \geq 2^i$ before multiplication. Let X_e be a random variable denoting the new weight of the edge e . Let Y_e be a random variable denoting the *relative weight* of edge e . The relative weight of an edge e at level i is $w(e)/2^i$. Let $w(e) = \beta 2^{i+1}$ where $1/2 \leq \beta \leq 1$. So X_e lies in the range $(\beta 2^{i+1}, \beta 2^{i+2})$. Y_e is defined as:

$$Y_e = \begin{cases} \frac{X_e}{2^i}, & \text{if } X_e \in (\beta 2^{i+1}, 2^{i+1}], \\ \frac{X_e}{2^{i+1}}, & \text{if } X_e \in (2^{i+1}, \beta 2^{i+2}). \end{cases}$$

Also, Y_e is a continuous random variable having probability density function given by : $f(x) = \frac{1}{\beta 2^{i+2} - \beta 2^{i+1}}$ for all $\beta 2^{i+1} \leq x \leq \beta 2^{i+2}$. The expected value of Y_e gives the expected value of γ . The expected value of Y_e is calculated as:

$$\begin{aligned}
E[Y_e] &= \frac{1}{\beta 2^{i+2} - \beta 2^{i+1}} \left(\int_{\beta 2^{i+1}}^{2^{i+1}} \frac{X}{2^i} dX \right) + \frac{1}{\beta 2^{i+2} - \beta 2^{i+1}} \left(\int_{2^{i+1}}^{\beta 2^{i+2}} \frac{X}{2^{i+1}} dX \right) \\
&= \frac{1}{\beta 2^{i+1}} \left(\frac{1}{2^i} \int_{\beta 2^{i+1}}^{2^{i+1}} X dX + \frac{1}{2^{i+1}} \int_{2^{i+1}}^{\beta 2^{i+2}} X dX \right) \\
&= \frac{1}{\beta 2^{i+1}} \left(\frac{1}{2^{i+1}} (X^2)^{2^{i+1}}_{\beta 2^{i+1}} + \frac{1}{2^{i+2}} (X^2)^{\beta 2^{i+2}}_{2^{i+1}} \right) \\
&= \frac{1}{\beta 2^{i+1}} \left(\frac{1}{2^{i+1}} ((2^{i+1})^2 - \beta^2 (2^{i+1})^2) + \frac{1}{2^{i+2}} (\beta^2 (2^{i+2})^2 - (2^{i+1})^2) \right) \\
&= \frac{1}{\beta} \left((1 - \beta^2) + (2\beta^2 - 1/2) \right) \\
&= \frac{1}{\beta} \left(\beta^2 + 1/2 \right)
\end{aligned}$$

The function $f(\beta) = \frac{1}{\beta} (\beta^2 + 1/2)$ has the maximum value when $\beta = 1$ and its maximum value is 1.5. From the above discussion, the expected value of $\gamma = E[Y_e]$. This implies an approximation ratio of $4E[Y_e] = 6$.

Theorem 3.1. *There exists a fully dynamic algorithm that maintains expected 6-MWM for any graph on n vertices in expected amortized $O(\log n \log C)$ time per update.*

Remark 3.1. *For each $1 \leq r \leq 2$, our algorithm computes a matching. This can be viewed as a mapping ϕ from $[1, 2]$ to the set of matchings computed by our algorithm. The above analysis says that there exists a r^* in $[1, 2]$ such that $\phi(r^*)$ is an 6-approximation of MWM of the given graph.*

Note that our randomization step finds a number from $[1, 2]$. We assumed that this random variable to be continuous over the above range. But in practice we only have $O(\log n)$ bits of randomness. In the appendix, we show that if we have $c \log n$ bits of randomness ($c > 1$), then we can get a approximation of $(6 + \epsilon)$ where $\epsilon = O(1/n^c)$.

4 Conclusion

We presented a fully dynamic algorithms for maintaining matching of large size or weight in graphs. The algorithm for maintaining 6-MWM is the first fully dynamic algorithm for maintaining approximate maximum weight matching.

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Appendix

We show that if we have $c \log n$ random bits ($c > 1$), we get $(6 + \epsilon)$ approximation of MWM. Assume that we get discrete random number between 0 and 1 where the gap between two discrete points is $1/n^c$. Consider the following function: $f(i) = i/n^c + 1$ for $0 \leq i \leq n^c$. The set $\{f(i) | 0 \leq i \leq n^c\}$ gives the range of discrete random variables generated in the range $[1, 2]$. Let e be an edge at level i having weight $w(e) = \beta 2^{i+1}$ where $1/2 \leq \beta \leq 1$. Let X_r denote the new weight of an edge e . Given a value of r , the value of X_r is $f(r) \cdot \beta 2^{i+1}$. When $X_r \leq 2^{i+1}$, $r \leq (\lfloor \frac{1}{\beta} \rfloor - 1)n^c$ and when $X_e = \beta 2^{i+2}$, $r = n^c$. Also note that a discrete random number $f(r)$ is generated with probability $1/n^c$. Remember that Y_e be a random variable denoting the *relative weight* of edge e . For the discrete case, we define it as follows:

$$Y_e = \begin{cases} \frac{X_r}{2^i}, & \text{if } 0 \leq r \leq (\lfloor \frac{1}{\beta} \rfloor - 1)n^c \\ \frac{X_r}{2^{i+1}}, & \text{if } (\lfloor \frac{1}{\beta} \rfloor - 1)n^c + 1 \leq r \leq n^c \end{cases}$$

Expected value of Y_e is calculated as:

$$\begin{aligned}
E[Y_e] &= \sum_{r=0}^{(\lfloor \frac{1}{\beta} \rfloor - 1)n^c} \frac{X_r}{2^i} \cdot \text{P}[f(r) \text{ generated}] + \sum_{r=(\lfloor \frac{1}{\beta} \rfloor - 1)n^c+1}^{n^c} \frac{X_r}{2^{i+1}} \cdot \text{P}[f(r) \text{ generated}] \\
&= \sum_{r=0}^{(\lfloor \frac{1}{\beta} \rfloor - 1)n^c} \frac{X_r}{2^i} \cdot \frac{1}{n^c} + \sum_{r=(\lfloor \frac{1}{\beta} \rfloor - 1)n^c+1}^{n^c} \frac{X_r}{2^{i+1}} \cdot \frac{1}{n^c} \\
&= \sum_{r=0}^{(\lfloor \frac{1}{\beta} \rfloor - 1)n^c} \frac{f(r)\beta 2^{i+1}}{2^i} \cdot \frac{1}{n^c} + \sum_{r=(\lfloor \frac{1}{\beta} \rfloor - 1)n^c+1}^{n^c} \frac{f(r)\beta 2^{i+1}}{2^{i+1}} \cdot \frac{1}{n^c} \\
&= \frac{\beta}{n^c} \left(2 \sum_{r=0}^{(\lfloor \frac{1}{\beta} \rfloor - 1)n^c} f(r) + \sum_{r=(\lfloor \frac{1}{\beta} \rfloor - 1)n^c+1}^{n^c} f(r) \right) \\
&= \frac{\beta}{n^c} \left(\sum_{r=0}^{n^c} (r/n^c + 1) + \sum_{r=0}^{(\lfloor \frac{1}{\beta} \rfloor - 1)n^c} (r/n^c + 1) \right) \\
&\leq \frac{\beta}{n^c} \left(\left(\frac{n^{2c}}{2n^c} + n^c + 1 \right) + \left(\frac{(\frac{1}{\beta} - 1)^2 n^{2c}}{2n^c} + (\frac{1}{\beta} - 1)n^c + 1 \right) \right) \\
&= \beta \left(\left(\frac{1}{2} + 1 + \frac{1}{n^c} \right) + \left(\frac{(\frac{1}{\beta} - 1)^2}{2} + (\frac{1}{\beta} - 1) + \frac{1}{n^c} \right) \right) \\
&= \beta \left(\left(\frac{3}{2} + \frac{1}{n^c} \right) + \left(\frac{1}{2\beta^2} - \frac{1}{2} + \frac{1}{n^c} \right) \right) \\
&= \beta \left(1 + \frac{1}{2\beta^2} + \frac{2}{n^c} \right)
\end{aligned}$$

$E[Y_e]$ has its maximum value when $\beta = 1$ and its maximum value is $1.5 + 2/n^c$. As discussed previously, the approximation ratio obtained by the algorithm is $4E[Y_e]$. This gives an approximation ratio of $6 + \epsilon$ where $\epsilon = 8/n^c$.